Exercise S12

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Continuum limit

$$Z = \sum_{j=1}^{n} g_j e^{-\beta E_j}$$

$$= \int_0^\infty dE\, g(E)\, e^{-\beta E}$$

$$g(E) = \gamma_s \frac{4\sqrt{2}\,\pi}{h^3} \pi V m^{\frac{3}{2}} E^{\frac{1}{2}}$$

γ_s = 2s+1 = 1 for spinless particle

$t = \beta E$, $dt = \beta dE$

$$Z = \int_0^\infty dE\, g_0 E^{\frac{1}{2}} e^{-\beta E} = g_0 \int_0^\infty dE\, E^{\frac{1}{2}} e^{-\beta E} = g_0 \frac{1}{\beta^{3/2}} \int_0^\infty dt\, t^{\frac{1}{2}} e^{-t}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

$$Z = \int_0^\infty dE\, g_0 E^{\frac{1}{2}} e^{-\beta E} = g_0 \int_0^\infty dE\, E^{\frac{1}{2}} e^{-\beta E} = g_0 2 \int_0^\infty ds\, s^2 e^{-\beta s^2}$$

$s^2 = E$

$2s\,ds = dE$

$$= 2g_0 \frac{\partial}{\partial -\beta} \int_0^\infty ds\, e^{-\beta s^2}$$

$\xi = \beta s^2$

$$= 2g_0 \frac{\partial}{\partial -\beta} \beta^{-\frac{1}{2}} \int_0^\infty ds\, \beta^{\frac{1}{2}} e^{-\xi^2}$$

$$= 2g_0 \frac{\partial}{\partial -\beta} \left(\beta^{-\frac{1}{2}} \int_0^\infty d\xi\, e^{-\xi^2}\right) = g_0 \frac{\partial}{\partial -\beta} \left(\beta^{-\frac{1}{2}} \sqrt{\pi}\right)$$

$$\int_0^\infty d\xi\, e^{-\xi^2} = \tfrac{1}{2}\sqrt{\pi}$$

$$= g_0 \frac{-\partial}{\partial \beta} \frac{\sqrt{\pi}}{\sqrt{\beta}} = g_0 \frac{1}{2} \frac{\sqrt{\pi}}{\beta^{\frac{3}{2}}}$$

$$= g_0 \tfrac{1}{2}\sqrt{\pi}\, (k_B T)^{\frac{3}{2}}$$

$$\langle U \rangle = \left\langle \sum_{j=1}^{n} N_j E_j \right\rangle$$

$$= \sum_{j=1}^{n} p_j N E_j$$

$$= N \sum_{j=1}^{n} p_j E_j$$

$$= N \langle E \rangle$$

$$= N \int_0^\infty dE\, E P(E)$$

$$= N \frac{\partial}{\partial -\beta} \ln Z$$

$$Z = Z_0 \beta^{-\frac{3}{2}}$$

$$= N \frac{-1}{Z} \frac{\partial Z}{\partial \beta}$$

$$= N \frac{-1}{Z_0} \beta^{\frac{3}{2}} \cdot -\frac{3}{2} \beta^{-\frac{5}{2}}$$

$$= N \frac{3}{2} \beta^{-1} = \frac{3}{2} N k_B T$$